Evaluating $\zeta(2)$

Robin Chapman
Department of Mathematics
University of Exeter, Exeter, EX4 4QE, UK
rjc@maths.ex.ac.uk

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I list several proofs of the celebrated identity:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad (1)$$

As it is clear that

$$\frac{3}{4} \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{m=1}^{\infty} \frac{1}{(2m)^2} = \sum_{r=0}^{\infty} \frac{1}{(2r+1)^2},$$

(1) is equivalent to

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}. \quad (2)$$

Many of the proofs establish this latter identity first.

None of these proofs is original; most are well known, but some are not as familiar as they might be. I shall try to assign credit the best I can, and I would be grateful to anyone who could shed light on the origin of any of these methods. I would like to thank Tony Lezard, José Carlos Santos and Ralph Krause, who spotted errors in earlier versions, and Richard Carr for pointing out an egregious solecism.

**Proof 1:** Note that

$$\frac{1}{n^2} = \int_{0}^{1} \int_{0}^{1} x^{n-1} y^{n-1} dx \, dy$$
and by the monotone convergence theorem we get
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \int_{0}^{1} \int_{0}^{1} \left( \sum_{n=1}^{\infty} (xy)^{n-1} \right) \, dx \, dy = \int_{0}^{1} \int_{0}^{1} \frac{dx \, dy}{1 - xy}. \]

We change variables in this by putting \((u, v) = ((x + y)/2, (y - x)/2)\), so that \((x, y) = (u - v, u + v)\). Hence
\[ \zeta(2) = 2 \int \int_{S} \frac{du \, dv}{1 - u^2 + v^2} \]
where \(S\) is the square with vertices \((0, 0), (1/2, -1/2), (1, 0)\) and \((1/2, 1/2)\). Exploiting the symmetry of the square we get
\[ \zeta(2) = 4 \int_{1/2}^{1} \int_{0}^{1} \frac{du \, dv}{1 - u^2 + v^2} + 4 \int_{1/2}^{1} \int_{0}^{1} \frac{dv \, du}{1 - u^2 + v^2} = 4 \int_{1/2}^{1} \frac{1}{\sqrt{1 - u^2}} \tan^{-1} \left( \frac{u}{\sqrt{1 - u^2}} \right) du \]
\[ + 4 \int_{1/2}^{1} \frac{1}{\sqrt{1 - u^2}} \tan^{-1} \left( \frac{1 - u}{\sqrt{1 - u^2}} \right) du. \]
Now \(\tan^{-1}(u/(\sqrt{1 - u^2})) = \sin^{-1} u\), and if \(\theta = \tan^{-1}((1 - u)/(\sqrt{1 - u^2}))\) then \(\tan^2 \theta = (1 - u)/(1 + u)\) and \(\sec^2 \theta = 2/(1 + u)\). It follows that \(u = 2 \cos^2 \theta - 1 = \cos 2\theta\) and so \(\theta = \frac{1}{2} \cos^{-1} u = \frac{\pi}{4} - \frac{1}{2} \sin^{-1} u\). Hence
\[ \zeta(2) = 4 \int_{0}^{1/2} \frac{\sin^{-1} u}{\sqrt{1 - u^2}} du + 4 \int_{1/2}^{1/2} \frac{1}{\sqrt{1 - u^2}} \left( \frac{\pi}{4} - \frac{\sin^{-1} u}{2} \right) du \]
\[ = \left[ 2(\sin^{-1} u)^2 \right]_{1/2}^{1/2} + \left[ \pi \sin^{-1} u - (\sin^{-1} u)^2 \right]_{1/2}^{1/2} \]
\[ = \frac{\pi^2}{18} + \frac{\pi^2}{2} - \frac{\pi^2}{4} - \frac{\pi^2}{6} + \frac{\pi^2}{36} = \frac{\pi^2}{6} \]
as required.

This is taken from an article in the *Mathematical Intelligencer* by Apostol in 1983.

**Proof 2:** We start in a similar fashion to Proof 1, but we use (2). We get
\[ \sum_{r=0}^{\infty} \frac{1}{(2r + 1)^2} = \int_{0}^{1} \int_{0}^{1} \frac{dx \, dy}{1 - x^2 y^2}. \]
We make the substitution

\[(u, v) = \left( \tan^{-1} x \sqrt{\frac{1 - y^2}{1 - x^2}}, \tan^{-1} y \sqrt{\frac{1 - x^2}{1 - y^2}} \right)\]

so that

\[(x, y) = \left( \frac{\sin u}{\cos v}, \frac{\sin v}{\cos u} \right) .\]

The Jacobian matrix is

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix}
\cos u / \cos v & \sin u \sin v / \cos^2 v \\
\sin u \sin v / \cos^2 u & \cos v / \cos u
\end{vmatrix}
\]

\[
= 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 u \cos^2 v}
\]

\[
= 1 - x^2y^2.
\]

Hence

\[
\frac{3}{4} \zeta(2) = \int \int_A du \, dv
\]

where

\[A = \{(u, v) : u > 0, v > 0, u + v < \pi/2\}\]

has area \(\pi^2/8\), and again we get \(\zeta(2) = \pi^2/6\).

This is due to Calabi, Beukers and Kock.

**Proof 3:** We use the power series for the inverse sine function:

\[
\sin^{-1} x = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n - 1) x^{2n+1}}{2 \cdot 4 \cdots (2n)} \frac{x^{2n+1}}{2n+1}
\]

valid for \(|x| \leq 1\). Putting \(x = \sin t\) we get

\[
t = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n - 1) \sin^{2n+1} t}{2 \cdot 4 \cdots 2n} \frac{\sin^{2n+1} t}{2n+1}
\]

for \(|t| \leq \frac{\pi}{2}\). Integrating from 0 to \(\frac{\pi}{2}\) and using the formula

\[
\int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)}
\]

gives us

\[
\frac{\pi^2}{8} = \int_0^{\pi/2} t \, dt = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}
\]
which is (2).

This comes from a note by Boo Rim Choe in the *American Mathematical Monthly* in 1987.

**Proof 4:** We use the $L^2$-completeness of the trigonometric functions. Let $e_n(x) = \exp(2\pi inx)$ where $n \in \mathbb{Z}$. The $e_n$ form a complete orthonormal set in $L^2[0, 1]$. If we denote the inner product in $L^2[0, 1]$ by $\langle \cdot, \cdot \rangle$, then Parseval’s formula states that

$$\langle f, f \rangle = \sum_{n=-\infty}^{\infty} |\langle f, e_n \rangle|^2$$

for all $f \in L^2[0, 1]$. We apply this to $f(x) = x$. We easily compute $\langle f, f \rangle = \frac{1}{3}$, $\langle f, e_0 \rangle = \frac{1}{2}$ and $\langle f, e_n \rangle = \frac{1}{2\pi in}$ for $n \neq 0$. Hence Parseval gives us

$$\frac{1}{3} = \frac{1}{4} + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{4\pi^2 n^2}$$

and so $\zeta(2) = \pi^2/6$.

Alternatively we can apply Parseval to $g = \chi_{[0,1/2]}$. We get $\langle g, g \rangle = \frac{1}{2}$, $\langle g, e_0 \rangle = \frac{1}{2}$ and $\langle g, e_n \rangle = ((-1)^n - 1)/2\pi in$ for $n \neq 0$. Hence Parseval gives us

$$\frac{1}{2} = \frac{1}{4} + \sum_{r=0}^{\infty} \frac{1}{\pi^2(2r + 1)^2}$$

and using (2) we again get $\zeta(2) = \pi^2/6$.

This is a textbook proof, found in many books on Fourier analysis.

**Proof 5:** We use the fact that if $f$ is continuous, of bounded variation on $[0, 1]$ and $f(0) = f(1)$, then the Fourier series of $f$ converges to $f$ pointwise. Applying this to $f(x) = x(1 - x)$ gives

$$x(1 - x) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{\pi^2 n^2},$$

and putting $x = 0$ we get $\zeta(2) = \pi^2/6$. Alternatively putting $x = 1/2$ gives

$$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which again is equivalent to $\zeta(2) = \pi^2/6$.

Another textbook proof.
**Proof 6:** Consider the series

\[ f(t) = \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}. \]

This is uniformly convergent on the real line. Now if \( \epsilon > 0 \), then for \( t \in [\epsilon, 2\pi - \epsilon] \) we have

\[
\sum_{n=1}^{N} \sin nt = \sum_{n=1}^{N} \frac{e^{int} - e^{-int}}{2i} = \frac{e^{it} - e^{i(N+1)t}}{2i(1 - e^{it})} - \frac{e^{-it} - e^{-i(N+1)t}}{2i(1 - e^{-it})} = \frac{e^{it} - e^{i(N+1)t}}{2i(1 - e^{it})} + \frac{1 - e^{-iNt}}{2i(1 - e^{it})}
\]

and so this sum is bounded above in absolute value by

\[
\frac{2}{|1 - e^{it}|} = \frac{1}{\sin t/2}.
\]

Hence these sums are uniformly bounded on \( [\epsilon, 2\pi - \epsilon] \) and by Dirichlet’s test the sum

\[ \sum_{n=1}^{\infty} \frac{\sin nt}{n} \]

is uniformly convergent on \( [\epsilon, 2\pi - \epsilon] \). It follows that for \( t \in (0, 2\pi) \)

\[
f'(t) = -\sum_{n=1}^{\infty} \frac{\sin nt}{n} = -\text{Im} \left( \sum_{n=1}^{\infty} \frac{e^{int}}{n} \right) = \text{Im}(\log(1 - e^{it})) = \arg(1 - e^{it}) = \frac{t - \pi}{2}.
\]

By the fundamental theorem of calculus we have

\[
f(\pi) - f(0) = \int_{0}^{\pi} \frac{t - \pi}{2} dt = -\frac{\pi^2}{4}.
\]

But \( f(0) = \zeta(2) \) and \( f(\pi) = \sum_{n=1}^{\infty} (-1)^n/n^2 = -\zeta(2)/2 \). Hence \( \zeta(2) = \pi^2/6 \).
Alternatively we can put

\[ D(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \]

the *dilogarithm* function. This is uniformly convergent on the closed unit disc, and satisfies \( D'(z) = -\frac{\log(1 - z)}{z} \) on the open unit disc. Note that \( f(t) = \text{Re} \, D(e^{2\pi i t}) \). We may now use arguments from complex variable theory to justify the above formula for \( f'(t) \).

This is just the previous proof with the Fourier theory eliminated.

**Proof 7:** We use the infinite product

\[ \sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \]

for the sine function. Comparing coefficients of \( x^3 \) in the Maclaurin series of sides immediately gives \( \zeta(2) = \frac{\pi^2}{6} \). An essentially equivalent proof comes from considering the coefficient of \( x \) in the formula

\[ \pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}. \]

The original proof of Euler!

**Proof 8:** We use the calculus of residues. Let \( f(z) = \pi z^{-2} \cot \pi z \). Then \( f \) has poles at precisely the integers; the pole at zero has residue \(-\pi^2/3\), and that at a non-zero integer \( n \) has residue \( 1/n^2 \). Let \( N \) be a natural number and let \( C_N \) be the square contour with vertices \((\pm 1 \pm i)(N + 1/2)\). By the calculus of residues

\[-\frac{\pi^2}{3} + 2 \sum_{n=1}^{N} \frac{1}{n^2} = \frac{1}{2\pi i} \int_{C_N} f(z) \, dz = I_N \]

say. Now if \( \pi z = x + iy \) a straightforward calculation yields

\[ |\cot \pi z|^2 = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y}. \]

It follows that if \( z \) lies on the vertical edges of \( C_n \) then

\[ |\cot \pi z|^2 = \frac{\sinh^2 y}{1 + \sinh^2 y} < 1 \]
and if \( z \) lies on the horizontal edges of \( C_n \)

\[
| \cot \pi z |^2 \leq \frac{1 + \sinh^2 \pi (N + 1/2)}{\sinh^2 \pi (N + 1/2)} = \coth^2 \pi (N + 1/2) \leq \coth^2 \pi / 2.
\]

Hence \( | \cot \pi z | \leq K = \coth \frac{\pi}{2} \) on \( C_N \), and so \( | f(z) | \leq \pi K / (N + 1/2)^2 \) on \( C_N \).

This estimate shows that

\[
| I_n | \leq \frac{1}{2\pi} \frac{\pi K}{(N + 1/2)^2} 8(N + 1/2)
\]

and so \( I_N \to 0 \) as \( N \to \infty \). Again we get \( \zeta(2) = \pi^2 / 6 \).

Another textbook proof, found in many books on complex analysis.

**Proof 9:** We first note that if \( 0 < x < \frac{\pi}{2} \) then \( \sin x < x < \tan x \) and so \( \cot^2 x < x^{-2} < 1 + \cot^2 x \). If \( n \) and \( N \) are natural numbers with \( 1 \leq n \leq N \) this implies that

\[
\cot^2 \frac{n\pi}{2N+1} < \frac{(2N+1)^2}{n^2\pi^2} < 1 + \cot^2 \frac{n\pi}{2N+1}
\]

and so

\[
\frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2 \frac{n\pi}{2N+1} < \sum_{n=1}^N \frac{1}{n^2}
\]

\[
< \frac{N\pi^2}{2N+1} + \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2 \frac{n\pi}{2N+1}.
\]

If

\[
A_N = \sum_{n=1}^N \cot^2 \frac{n\pi}{2N+1}
\]

it suffices to show that \( \lim_{N \to \infty} A_N / N^2 = \frac{2}{3} \).

If \( 1 \leq n \leq N \) and \( \theta = n\pi / (2N+1) \), then \( \sin(2N+1)\theta = 0 \) but \( \sin \theta \neq 0 \). Now \( \sin(2N+1)\theta \) is the imaginary part of \( (\cos \theta + i \sin \theta)^{2N+1} \), and so

\[
\frac{\sin(2N+1)\theta}{\sin^{2N+1} \theta} = \frac{1}{\sin^{2N+1} \theta} \sum_{k=0}^N (-1)^k \binom{2N+1}{2N-2k} \cos^{2(N-k)} \theta \sin^{2k+1} \theta
\]

\[
= \sum_{k=0}^N (-1)^k \binom{2N+1}{2N-2k} \cot^{2(N-k)} \theta
\]

\[
= f(\cot^2 \theta)
\]
say, where \( f(x) = (2N+1)x^N - (\frac{2N+1}{3})x^{N-1} + \cdots \). Hence the roots of \( f(x) = 0 \) are \( \cot^2(n\pi/(2N+1)) \) where \( 1 \leq n \leq N \) and so \( A_N = N(2N-1)/3 \). Thus \( A_N/N^2 \to \frac{2}{3} \), as required.

This is an exercise in Apostol’s *Mathematical Analysis* (Addison-Wesley, 1974).

**Proof 10:** Given an odd integer \( n = 2m + 1 \) it is well known that \( \sin nx = F_n(\sin x) \) where \( F_n \) is a polynomial of degree \( n \). Since the zeros of \( F_n(y) \) are the values \( \sin(j\pi/n) \) \((-m \leq j \leq m)\) and \( \lim_{y \to 0}(F_n(y)/y) = n \) then

\[
F_n(y) = ny \prod_{j=1}^{m} \left(1 - \frac{y^2}{\sin^2(j\pi/n)}\right)
\]

and so

\[
\sin nx = n \sin x \prod_{j=1}^{m} \left(1 - \frac{\sin^2 x}{\sin^2(j\pi/n)}\right).
\]

Comparing the coefficients of \( x^3 \) in the MacLaurin expansion of both sides gives

\[
-\frac{n^3}{6} = -\frac{n}{6} - n \sum_{j=1}^{m} \frac{1}{\sin^2(j\pi/n)}
\]

and so

\[
\frac{1}{6} - \sum_{j=1}^{m} \frac{1}{n^2 \sin^2(j\pi/n)} = \frac{1}{6n^2}.
\]

Fix an integer \( M \) and let \( m > M \). Then

\[
\frac{1}{6} - \sum_{j=1}^{M} \frac{1}{n^2 \sin^2(j\pi/n)} = \frac{1}{6n^2} + \sum_{j=M+1}^{m} \frac{1}{n^2 \sin^2(j\pi/n)}
\]

and using the inequality \( \sin x > \frac{2}{\pi}x \) for \( 0 < x < \frac{\pi}{2} \), we get

\[
0 < \frac{1}{6} - \sum_{j=1}^{M} \frac{1}{n^2 \sin^2(j\pi/n)} < \frac{1}{6n^2} + \sum_{j=M+1}^{m} \frac{1}{4j^2}.
\]

Letting \( m \) tend to infinity now gives

\[
0 \leq \frac{1}{6} - \sum_{j=1}^{M} \frac{1}{\pi^2 j^2} \leq \sum_{j=M+1}^{\infty} \frac{1}{4j^2}.
\]
Hence
\[ \sum_{j=1}^{\infty} \frac{1}{\pi^2 j^2} = \frac{1}{6}. \]

This comes from a note by Kortram in *Mathematics Magazine* in 1996.

**Proof 11:** Consider the integrals
\[ I_n = \int_{0}^{\pi/2} \cos^{2n} x \, dx \quad \text{and} \quad J_n = \int_{0}^{\pi/2} x^2 \cos^{2n} x \, dx. \]

By a well-known reduction formula
\[ I_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1) \pi}{2 \cdot 4 \cdot 6 \cdots 2n} = \frac{(2n)! \pi}{4^n n!^2 2}. \]

If \( n > 0 \) then integration by parts gives
\[
I_n = \left[ x \cos^{2n} x \right]_{0}^{\pi/2} + 2n \int_{0}^{\pi/2} x \sin x \cos^{2n-1} x \, dx \\
= n \left[ x^2 \sin x \cos^{2n-1} x \right]_{0}^{\pi/2} \\
- n \int_{0}^{\pi/2} x^2 (\cos^{2n} x - (2n - 1) \sin^2 x \cos^{2n-2} x) \, dx \\
= n(2n - 1) J_{n-1} - 2n^2 J_n.
\]

Hence
\[ \frac{(2n)! \pi}{4^n n!^2 2} = n(2n - 1) J_{n-1} - 2n^2 J_n \]
and so
\[ \frac{\pi}{4n^2} = \frac{4^{n-1} (n-1)!^2}{(2n-2)!} J_{n-1} - \frac{4^n n!^2}{(2n)!} J_n. \]

Adding this up from \( n = 1 \) to \( N \) gives
\[ \frac{\pi}{4} \sum_{n=1}^{N} \frac{1}{n^2} = J_0 - \frac{4^N N!^2}{(2N)!} J_N. \]

Since \( J_0 = \pi^3/24 \) it suffices to show that \( \lim_{N \to \infty} 4^N N!^2 J_N/(2N)! = 0 \). But the inequality \( x < \frac{\pi}{2} \sin x \) for \( 0 < x < \frac{\pi}{2} \) gives
\[ J_N < \frac{\pi^2}{4} \int_{0}^{\pi/2} \sin^2 x \cos^{2N} x \, dx = \frac{\pi^2}{4} (I_N - I_{N+1}) = \frac{\pi^2 I_N}{8(N+1)} \]
and so
\[ 0 < \frac{4^N N!}{(2N)!} J_N < \frac{\pi^3}{16(N + 1)}. \]
This completes the proof.

This proof is due to Matsuoka (American Mathematical Monthly, 1961).

**Proof 12:** Consider the well-known identity for the Fejér kernel:
\[
\left( \frac{\sin n x/2}{\sin x/2} \right)^2 = \sum_{k=-n}^{n} (n - |k|) e^{ikx} = n + 2 \sum_{k=1}^{n} (n - k) \cos kx.
\]
Hence
\[
\int_0^\pi x \left( \frac{\sin n x/2}{\sin x/2} \right)^2 \, dx = \frac{n \pi^2}{2} + 2 \sum_{k=1}^{n} (n - k) \int_0^\pi x \cos kx \, dx
\]
\[
\quad = \frac{n \pi^2}{2} - 2 \sum_{k=1}^{n} (n - k) \frac{1 - (-1)^k}{k^2}
\]
\[
\quad = \frac{n \pi^2}{2} - 4n \sum_{1 \leq k \leq n, 2k} \frac{1}{k^2} + 4 \sum_{1 \leq k \leq n, 2k} \frac{1}{k}
\]
If we let \( n = 2N \) with \( N \) an integer then
\[
\int_0^\pi x \frac{\sin N x}{\sin x/2}^2 \, dx = \frac{\pi^2}{8} - \sum_{r=0}^{N-1} \frac{1}{(2r + 1)^2} + O \left( \frac{\log N}{N} \right).
\]
But since \( \sin \frac{x}{2} > \frac{x}{\pi} \) for \( 0 < x < \pi \) then
\[
\int_0^\pi x \frac{\sin N x}{\sin x/2}^2 \, dx < \frac{\pi^2}{8N} \int_0^\pi \sin^2 N x \, dx
\]
\[
\quad = \frac{\pi^2}{8N} \int_0^{N\pi} \sin^2 y \, dy = O \left( \frac{\log N}{N} \right)
\]
Taking limits as \( N \to \infty \) gives
\[
\frac{\pi^2}{8} = \sum_{r=0}^{\infty} \frac{1}{(2r + 1)^2}.
\]
This proof is due to Stark (American Mathematical Monthly, 1969).

**Proof 13:** We carefully square Gregory’s formula
\[
\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}.
\]
We can rewrite this as \( \lim_{N \to \infty} a_N = \frac{\pi}{2} \) where
\[
a_N = \sum_{n=-N}^{N} \frac{(-1)^n}{2n+1}.
\]
Let
\[
b_N = \sum_{n=-N}^{N} \frac{1}{(2n+1)^2}.
\]
By (2) it suffices to show that \( \lim_{N \to \infty} b_N = \pi^2/4 \), so we shall show that \( \lim_{N \to \infty} (a_N^2 - b_N) = 0 \).

If \( n \neq m \) then
\[
\frac{1}{(2n+1)(2m+1)} = \frac{1}{2(m-n)} \left( \frac{1}{2n+1} - \frac{1}{2m+1} \right)
\]
and so
\[
a_N^2 - b_N = \sum_{n=-N}^{N} \sum_{m=-N}^{N} \frac{(-1)^{m+n}}{2(m-n)} \left( \frac{1}{2n+1} - \frac{1}{2m+1} \right)
\]
\[
= \sum_{n=-N}^{N} \sum_{m=-N}^{N} (-1)^{m+n} \frac{1}{(2n+1)(m-n)}
\]
\[
= \sum_{n=-N}^{N} \frac{(-1)^n c_{n,N}}{2n+1}
\]
where the dash on the summations means that terms with zero denominators are omitted, and
\[
c_{n,N} = \sum_{m=-N}^{N} (-1)^m \frac{1}{(m-n)}.
\]
It is easy to see that \( c_{-n,N} = -c_{n,N} \) and so \( c_{0,N} = 0 \). If \( n > 0 \) then
\[
c_{n,N} = (-1)^{n+1} \sum_{j=N-n+1}^{N+n} \frac{(-1)^j}{j}
\]
and so \( |c_{n,N}| \leq 1/(N-n+1) \) as the magnitude of this alternating sum is not more than that of its first term. Thus
\[
|a_N^2 - b_N| \leq \sum_{n=1}^{N} \left( \frac{1}{(2n-1)(N-n+1)} + \frac{1}{(2n+1)(N-n+1)} \right)
\]

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\[
\begin{align*}
&= \sum_{n=1}^{N} \frac{1}{2N+1} \left( \frac{2}{2n-1} + \frac{1}{N-n+1} \right) \\
&\quad + \sum_{n=1}^{N} \frac{1}{2N+3} \left( \frac{2}{2n+1} + \frac{1}{N-n+1} \right) \\
&\leq \frac{1}{2N+1} (2 + 4\log(2N+1) + 2 + 2\log(N+1))
\end{align*}
\]

and so \( a_N^2 - b_N \to 0 \) as \( N \to \infty \) as required.

This is an exercise in Borwein & Borwein’s *Pi and the AGM* (Wiley, 1987).

**Proof 14:** This depends on the formula for the number of representations of a positive integer as a sum of four squares. Let \( r(n) \) be the number of quadruples \((x, y, z, t)\) of integers such that \( n = x^2 + y^2 + z^2 + t^2 \). Trivially \( r(0) = 1 \) and it is well known that

\[
r(n) = 8 \sum_{m|n, 4m} m
\]

for \( n > 0 \). Let \( R(N) = \sum_{n=0}^{N} r(n) \). It is easy to see that \( R(N) \) is asymptotic to the volume of the 4-dimensional ball of radius \( \sqrt{N} \), i.e., \( R(N) \sim \frac{\pi^2}{2} N^2 \).

But

\[
R(N) = 1 + 8 \sum_{n=1}^{N} \sum_{m|n, 4m} m = 1 + 8 \sum_{m\leq N, 4m} m \left\lfloor \frac{N}{m} \right\rfloor = 1 + 8(\theta(N) - 4\theta(N/4))
\]

where

\[
\theta(x) = \sum_{m\leq x} m \left\lfloor \frac{x}{m} \right\rfloor.
\]

But

\[
\theta(x) = \sum_{mr \leq x} m \\
= \sum_{r \leq x} \sum_{m=1}^{\left\lfloor x/r \right\rfloor} m \\
= \frac{1}{2} \sum_{r \leq x} \left( \left\lfloor \frac{x}{r} \right\rfloor^2 + \left\lfloor \frac{x}{r} \right\rfloor \right) \\
= \frac{1}{2} \sum_{r \leq x} \left( \frac{x^2}{r^2} + O\left(\frac{x}{r}\right) \right)
\]
\[
\frac{x^2}{2} (\zeta(2) + O(1/x)) + O(x \log x)
\]
\[
= \frac{\zeta(2)x^2}{2} + O(x \log x)
\]
as \(x \to \infty\). Hence

\[
R(N) \sim \frac{\pi^2}{2} N^2 \sim 4\zeta(2) \left( N^2 - \frac{N^2}{4} \right)
\]
and so \(\zeta(2) = \pi^2/6\).

This is an exercise in Hua’s textbook on number theory.